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COMMENT

Exact analytical solutions for the cut-off Coulomb potential $V(r) = -Ze^2/(r+\beta)$

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Abstract. We derive exact analytical solutions for the cut-off Coulomb potential $V(r) = -Ze^2/(r+\beta)$. The solutions are derived using the known results of supersymmetric quantum mechanics and it is found that exact solutions exist when β and l satisfy a supersymmetric constraint. Our results have been compared whenever possible with the published $1/N$-shifted expansion results.

Of late the formalism of supersymmetric quantum mechanics (SUSYQM) has been employed to determine exact analytical solutions of the Schrödinger equation both in one and three dimensions (1)-(3). The method depends on the construction of a superpotential so that the SUSY potential can be compared with the potential whose solution is sought. In this comment we would find exact solutions for the cut-off Coulomb potential $V(r) = -Ze^2/(r+\beta)$, $\beta > 0$ both for $l = 0$ and $l \neq 0$ when β and l satisfy some constraints. This potential has been taken as an approximation due to a smeared charge distribution and was considered for describing mesonic atoms [5]. There have been numerical studies, including a shifted $1/N$ expansion to calculate the eigenvalues for this potential [6], but so far as we know an exact solution for a wide range of values of l has not been presented before. Before applying SUSYQM we recall that in one dimension a SUSYQM Hamiltonian consists of a pair of Hamiltonians [7],

$$H = \{Q^+, Q\}. \tag{1}$$

Using explicit expressions for Q and Q^+ [1], H can be written as

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{2}$$

where

$$H_+ = -\frac{d^2}{dx^2} + V_+(x) \tag{3}$$

$$H_- = -\frac{d^2}{dx^2} + V_-(x) \tag{4}$$

with

$$V_\pm(x) = W^2(x) \pm W'(x). \tag{5}$$

The wavefunctions on which H operates are two-component column vectors of the form

$$\varphi(x)=\begin{pmatrix}\varphi_+(x)\\ \varphi_-(x)\end{pmatrix}. \tag{6}$$

It may be pointed out that the ground state $|\phi^0\rangle$ is always annihilated by the supercharges:

$$Q|\varphi^0\rangle = Q^+|\varphi^0\rangle = 0. \tag{7}$$

From (7) it follows that the ground state wavefunctions are of the form

$$\begin{pmatrix}\varphi_+^0(x)\\ 0\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}0\\ \varphi_-^0(x)\end{pmatrix} \tag{8}$$

where

$$\varphi_\pm^0(x)\sim\exp\left(\pm\int^x W(t)\,dt\right). \tag{8'}$$

If either of $\varphi_\pm^0(x)$ is normalisable then supersymmetry is unbroken and the ground state energy is zero. Let us now turn to the choice of the superpotential. Now the radial Schrödinger equation for any radially symmetric potential $V(r)$ can be written as (after multiplying both sides by two and taking $\hbar=m=c=1$),

$$-\left(\frac{d^2\psi}{dr^2}+\frac{2}{r}\frac{d\psi}{dr}\right)+2V_E(r)=2E\psi(r). \tag{9}$$

Putting $\psi=r^{-1}\phi(r)$ this can be written as ($V_E=V+[l(l+1)/2r^2]$)

$$\frac{d^2\varphi}{dr^2}+[2E-2V_E(r)]\varphi(r)=0. \tag{10}$$

For $V(r)=-Ze^2/(r+\beta)$ we make the following ansatz for the superpotential

$$W(r)=\frac{C}{r}+\frac{1}{r+\beta}+\sum_{i=1}^{N}\frac{1}{r-g_i}+b. \tag{11}$$

We now identify the (+) sector (i.e. W^2+W') with the effective potential term in (9).

First consider the case $N=0$

$$V_+(r)=\frac{C(C-1)}{r^2}+\frac{2C}{r}\left(b+\frac{1}{\beta}\right)+\frac{2}{r+\beta}\left(b-\frac{C}{\beta}\right)+b^2. \tag{12}$$

If we now set

$$V_+(r)-E_+=2V(r)+\frac{l(l+1)}{r^2}-2E \tag{13}$$

we must have

$$C(C-1)=l(l+1) \tag{14}$$

$$b=-1/\beta \tag{15}$$

$$b-C/\beta=-Ze^2 \tag{16}$$

and

$$2E=E_+-b^2. \tag{17}$$

From (14) $C = -l$ or $l+1$. We take $C = l+1$ to make $\exp\int[w(r)\,dz]$ normalisable. Then solving (15), (16) and (17) we get

$$\beta = \frac{l+2}{Ze^2} \qquad \text{and} \qquad E = -\frac{1}{2\beta^2} \tag{18}$$

where we have taken $E_+ = 0$, for the ground state supersymmetric energy. The normalised eigenfunction for the above case is given by

$$\psi(r) = Nr^{l+1}(r+\beta)\,e^{-r/\beta} \tag{19}$$

where

$$N^{-1} = [\beta^{2l+5}(2l+4)(2l+7)\Gamma(2l+3)/2^{2l+5}]^{1/2}. \tag{20}$$

Some numerical results are given in table 1 to compare our results with the $1/N$-shifted expansion result (5) whenever possible.

Next consider the case $N = 1$. Here

$$W = \frac{C}{r} + \frac{1}{r+\beta} + \frac{1}{r-g_1} + b. \tag{21}$$

Proceeding as before we obtain

$$C = l+1 \tag{22}$$

$$g_1 = \left(\frac{1}{\beta} - \frac{Ze^2}{l+3}\right)^{-1} \tag{23}$$

$$b = -\frac{Ze^2}{l+3} \tag{24}$$

and

$$E = -\frac{1}{2}\left(\frac{Ze^2}{l+3}\right)^2 \tag{25}$$

and the relation between β and l is given by

$$\beta = \frac{(3l^2+15l+18) \pm [(3l^2+15l+18)^2 - 4(l+2)(2l^3+15l^2+36l+27)]^{1/2}}{2(l+2)}. \tag{26}$$

In table 2 we give some numerical results for positive g_1. A positive value of g_1 gives a node in the wavefunction and hence gives the results for the first excited states.

Table 1. Some numerical results. E^* are the exact supersymmetric results, and E^+ are the $1/N$-shifted expansion results obtained from [5].

l	β	E^*	E^+
0	2	−0.12500	−0.124858
1	3	−0.05555	−0.05554
2	4	−0.03125	−0.03125
3	5	−0.02000	−0.019199
4	6	−0.01388	
5	7	−0.01020	

Table 2. Some numerical results for positive g_1.

l	E	β	g_1
1	−0.03125 (−0.03105)[a]	2.945	11.166
2	−0.02000 (−0.01995)[b]	3.964	19.142
3	−0.01388	4.975	29.125
4	−0.00102	5.982	41.111
5	−0.00781	6.986	55.100

[a] The $1/N$-shifted expansion value for $\beta = 3$.
[b] The $1/N$-shifted value for $\beta = 4$.

The eigenfunction for the above solution is given by $(b<0)$

$$\psi = Nr^{l+1}(r+\beta)(r-g_1)\,\mathrm{e}^{br} \tag{27}$$

where

$$N^{-1} = \{(-2b)^{-(2l+7)}[\Gamma(2l+7) - 4b(\beta-g_1)\Gamma(2l+6) + 4b^2(\beta^2 - 4\beta g_1 + g_1^2) \times\Gamma(2l+5) - 16b^3 g_1(g_1-\beta)\beta\Gamma(2l+4) + 16b^4 g_1^2\beta^2\Gamma(2l+3)]\}^{1/2}. \tag{28}$$

For the general case when $W(r)$ is given by (11) the parameters C, g_i and b are given by

$$C = l+1 \tag{29}$$

$$b + \frac{1}{\beta} - \sum_{i=1}^{N} \frac{1}{g_i} = 0 \tag{30}$$

$$b - \frac{C}{\beta} - \sum_{i=1}^{N} \frac{1}{\beta + g_i} = -Ze^2 \tag{31}$$

$$b + \frac{C}{g_i} + \frac{1}{g_i+\beta} + \sum_{j\neq i} \frac{1}{g_i - g_j} = 0. \tag{32}$$

The corresponding wavefunction is given by

$$\psi = Nr^{l+1}(r+\beta)\,\mathrm{e}^{br} \prod_{i=1}^{N} (r-g_i) \tag{33}$$

where N is the normalisation constant. Thus, in principle, one can obtain an arbitrary number of exact solutions by solving the above equations. Solutions obtained by the numerical perturbation method can always be checked against these exact solutions.

References

[1] Roy P and Roychoudhury R 1987 *Phys. Lett.* A **122** 275
[2] Roychoudhury R and Varshni Y P 1988 *Phys. Rev.* A **37** 2307
[3] Roy P and Roychoudhury R 1987 *J. Phys. A: Math. Gen.* **20** 6597
[4] Adhikari R, Dutt R and Varshni Y P 1989 *Phys. Lett.* A **141** 1
[5] Mehta C H and Patil S H 1978 *Phys. Rev.* A **17** 43
[6] Ray P P and Mahata K 1989 *J. Phys. A: Math. Gen.* **22** 3161
[7] Cooper F and Freedman B 1983 *Ann. Phys., NY* **146** 262